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A mathematical model of psychotherapy: An investigation using dynamic non-linear equations to model the therapeutic relationship

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Abstract
Mathematical models, such as the one developed by Gottman et al. (1998, 2000, 2002) to understand the interaction between husbands and wives, can provide novel insights into the dynamics of the therapeutic relationship. A set of nonlinear equations were used to model the changing emotional state of a therapist and client. The results suggest: (1) The person that is most responsive to the other achieves the most positive state, (2) the emotional state of the client oscillates before reaching its final state, (3) therapy is least successful when the therapist starts from a negative state, and (4) there is an inverse relationship between models that change only the influence parameter and models that change only the inertia parameter, creating a series of four basic models to work with. These theoretical models require further, empirical investigation to test the derived parameters. If validated, or revised based on observations of therapist-client relationships in development, they could provide specific direction in creating successful therapeutic relationships for training clinicians and those already in practice.

Keywords: dynamical systems; alliance; emotion in therapy; process research; psychotherapist training/supervision/development; statistical methodology; technology in psychotherapy research and training

The rates of psychological disorders are staggering. According to the National Institute of Mental Health (NIMH), one in four adults in the United States suffers with a diagnosable mental illness (NIMH, 2010). These disorders exact a physical, emotional and economic toll on the individuals afflicted with them, their families, their communities, and the nation as a whole. They are the leading cause of disabilities in the United States. Psychotherapy has been shown to be an effective method for treating a wide array of mental illness (Kazdin, 2008; Lambert & Barley, 2002; Mozdzierz, Peluso, & Lisecki, 2009). However, only 25% of individuals who do have a diagnosable mental illness will come to therapy; and of those, approximately half will drop out after the first session (Muran, Gorman, Eubanks-Carter, et al., 2009). This rate of attrition has remained consistent over the last 50 years (Barrett, Chua, Crits-Christoph, Gibbons, & Thompson, 2008). The major question that has confronted clinicians and researchers alike continues to be: if therapy is successful and effective, then why don't clients stay?

Lambert and Barley (2002) summarized the available research and concluded that certain variables contributed significantly different percentages to successful therapeutic outcomes. They found that extratherapeutic change (defined as client characteristics outside the therapeutic interaction) contributes 40% towards the effectiveness of therapy; "common factors" (that is, elements common to all therapies) contribute 30%; expectancy (hope, "placebo effects," etc.) contributes 15%; and techniques or theoretical approaches contribute 15%. Lambert and Barley carefully point out that the data accumulated from psychotherapy research demonstrate that common, interpersonal factors contribute more to therapeutic outcome than specific techniques.

The literature has traditionally tried to describe the salient qualities of the "common factors" in therapy by referring to *therapeutic relationship* (Gelso, 2009; Norcross, 2010; Norcross & Wampold, 2011). Indeed, without a positive therapeutic relationship between client and counselor, nothing significant would be accomplished in therapy. Norcross (2002),

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reporting on Division 29 of the American Psychological Association's Task Force on Empirically Supported Therapy Relationships, defined the psychotherapy relationship as: "the feelings and attitudes that therapist and client have toward one another, *and the manner in which these are expressed"* (p. 7; emphasis added). The therapeutic relationship is an essential element in counseling therapy clients. The quality of the therapeutic relationship is one of the most potent and lasting indicators of successful treatment. However, the qualities of this relationship and the interaction between the therapist and client that are most effective to help change negative behaviors or habits are not yet well understood (Kazdin, 2008; Gelso, 2009; Norcross & Wampold, 2011).

The Therapeutic Relationship

Even though it is a professional engagement between a client and a practitioner, at its core the therapeutic relationship is essentially a human encounter. It is the person of the therapist that the client experiences, evaluates, and reacts to in treatment seemingly no matter what sort of treatment, theory, or technique the therapist espouses to practice (Orlinsky & Howard, 1977). Orlinsky, Grave, and Parks (1994) reviewed more than 2000 process-outcome studies since 1950, and identified several therapist variables that were consistently shown to have a positive impact on treatment outcome: "Therapist credibility, skill, empathic understanding, and affirmation of the patient, along with the ability to engage with the patient, to focus on the patient's problems, and to direct the patient's attention to the patient's affective experience, were highly related to successful treatment" (cited in Lambert & Barley, 2002, p. 22).

Centorrino et al. (2001) demonstrated how important the therapeutic relationship is in determining successful treatment outcomes. They investigated factors associated with outpatient mental health treatment compliance (i.e., keeping scheduled clinic appointments) versus noncompliance (i.e., failure to keep appointments and treatment drop-outs). Only three factors contributed to treatment compliance: (1) The perceived warmth and friendliness of the therapist; (2) talking to the client about something that was of importance to the client; and (3) talking to the client in a structured manner. These three relational factors were shown to be more important in determining outcome than client diagnosis or demographics. Lastly, clients who felt that they were going to be listened to by a therapist, rather than merely treated by a medical professional with medications, were more likely to be compliant (a prerequisite for eventual success in therapy).

In their meta-analysis of 79 studies of psychotherapy outcome, Martin, Garske, and Davis (2000) determined that the connection between the therapeutic relationship (and a specific element of the relationship, the "working alliance") and outcomes is consistent. In addition, Martin et al. suggest that there is a therapeutic/healing effect in the relationship itself, and that if an appropriate alliance is established, a client will experience the relationship as therapeutic regardless of other psychological interventions. Martin et al. found that other variables do not seem to influence the strength/quality of relationship, and that the finding was consistent regardless of the type of measure that is used; when the alliance assessment is taken (early or late in therapy); and type of treatment provided. Horvath and Bedi (2002) summarized 90 outcome studies conducted between 1976 and 2000 and concluded that clients who were aligned with their therapist had positive outcomes. More recently, Muran et al. (2009) and Gelso (2009) concluded that client and therapist ratings of the strength of the therapeutic alliance consistently predicted success in treatment. Lastly, Horvath, Del Re, Fluckiger, and Symonds (2011) followed up their previous research with an analysis that included twice the number of studies as before, and found a similar (though slightly higher) effect size ($r = .275$) between therapeutic outcome and the strength of the alliance.

The Real Relationship

It is evident from the empirical evidence that a strong affective bond resulting in a consistent therapeutic alliance is an important vehicle in determining successful treatment outcomes (Mozdzierz, Peluso & Lisecki, 2009). However, given the high number of rates of premature dropout in therapy, there is an element that seems to be missing. Over 30 years ago, Greenson (1965, 1967), as cited in Gelso, 2009) delineated between the professional elements of the therapeutic encounter (therapeutic relationship, the working alliance, and the issues of transference and countertransferrence) and the personal encounter. He believed that this personal encounter, called the *real* relationship, was fundamental to the success of the therapeutic encounter, and that these overlapping elements are often difficult to distinguish from one another (as cited in Gelso, 2009).

While they may seem indistinguishable, according to Gelso (2009), the working alliance and the real relationship are different in that "(w)hereas the working alliance is an artifact of the treatment in the sense that its only reason for existence is to get the work done, the real relationship exists to one degree or another any time two or more people relate to one another" (p. 257). Indeed, Greenson, Gelso

and others define the real relationship as one that contains genuineness, realism, and (positive) emotional valence. It is an emergent property of any two people coming together for mutual benefit. As a result, both client and therapist contribute to creating this element of the therapeutic encounter.

A key element to the real relationship is valence, or how positive or negative the individuals' attitudes and feelings are toward one another. It is a critical element of the development of the real relationship (and thus, the working alliance). According to Gelso, discussing the importance of the emotional valance of the therapeutic relationship is not traditionally held in as high esteem as the concepts of therapist neutrality and distance. Nonetheless, the therapeutic relationship, by virtue of the real relationship, is not unlike any other human relationship (e.g., marriage, friendship, etc.) between two people, and is highly dependent on the valence of relationship between client and therapist.

Clearly, one of the most powerful mechanisms of change between a therapist and a client is the therapeutic relationship. However, the essentials of how to create this relationship—the interaction between client and therapist—have not been modeled or observed in a direct fashion. Kazdin (2008) advocated that researchers should, "attempt to understand more about the many change processes and how they can be triggered, activated, exploited, and trained" (p. 157). According to Kazdin, merely researching the strength of the alliance is not enough. Instead, intervening process must be studied that can show the mechanisms of therapy at work. These studies must include a demonstration of the timeline (e.g., showing cause and effect of the therapeutic process on a client), and an explanation of how the therapeutic process works to improve client functioning and lead to positive treatment outcomes (Kazdin, 2008). Norcross and Wampold (2011) highlight the concept of therapist *responsiveness* to a client's needs as both a particular strength of the therapeutic alliance, and a difficult construct to quantify and measure:

Effective psychotherapists are responsive to the different needs of their clients in different cases, and within same case, at different moments. Successful responsiveness can confound attempts to find naturalistically observed linear relations of outcomes with therapist behaviors (e.g., cohesion, positive regard). Because of such problems, the statistical relations between the relationship and outcome cannot always be trusted. By being clinically attuned and flexible, psychotherapists make it more difficult in research studies to discern what works (p. 100).

We believe that a dynamical systems approach that uses non-linear differential equations can provide a method for modeling some of these complex relational dynamics, and provide information on the nature and composition of the working alliance, and the impact that the emotional valence within the relationship (viz. the "real relationship") has on the successful progress of therapy (or the lack therof) that has remained elusive.

Mathematical Models: Using Dynamic Nonlinear Equations to Model Human Relationships

The most common paradigms used in social science to analyze experimental data are to determine the functional relationships between dependent and independent variables, the correlations between different variables, or the most meaningful factors that account for the data as determined by their statistical properties (e.g., by ANOVA). By contrast, in a recent issue of *American Psychologist*, Vallacher, Coleman, Nowak, and Bui-Wrzosinska (2010) advocated the usefulness of dynamical systems for the modeling and study of complex phenomena using simple linear formulae. Dynamic nonlinear mathematical equations allow for investigators to understand complex systems that are apt to change (as with the "responsiveness" that Norcross and Wampold described above). The ultimate goal in creating and using these equations is to define the dynamics of a system and evaluate how the system changes over time (Gottman, Murray, Swanson, Tyson, & Swanson, 2002). This approach is common in the *physical sciences*, but rarely used in the *social sciences* (Gottman & Notarius, 2000, 2002), though this is beginning to change (see Vallacher et al., 2010). A dynamical systems approach does not to seek correlations between variables, but: (1) uses experience, data, and intuition as the starting point to develop a mathematical model, (2) uses mathematics to "solve" the model, that its to determine its mathematical properties, and (3) then uses those properties to make predictions which can be empirically tested. Over the last 15 years, dynamical systems approaches have begun to be used in social, developmental and clinical psychology to describe human systems. Granic and her colleagues (Granic & Hollenstein, 2003; Granic & Lamey, 2002) have used dynamical systems to successfully model parent-child interactions and discover attractors in the system and other relational patterns that could guide intervention. Gardener, Burr and Wiedower (2006) have even suggested that dynamical systems may be able to provide some traditional schools of family therapy that were based on systems theory with a mathematical tool to explain various theoretical components (e.g., homeostasis) as well as some

quintessential family therapy techniques (e.g., paradoxical interventions, circular questioning, etc.).

Gottman et al.'s Work

Gottman and his associates (Cook et al., 1995; Gottman et al., 2002; Gottman, Coan, Carrere, & Swanson, 1998) have used dynamic nonlinear equations to model the interaction between husbands and wives in relation to each other. Specifically, they utilized this quantitative mathematical approach to be able to understand the changes in the marital dyad between the members of the couple, and the relative influence that each has on the other (Gottman et al., 1998). These equations (one for the wife, one for the husband) allowed the researchers to model both the positive and negative interactions of the couple. They can be used to determine the stable "steady states" or points of homeostasis within the system (i.e., the relationship). These steady states function as an anchor point that brings the system back to homeostasis if the system is perturbed, or moved by a force away from homeostasis. In regulated systems, these steady states can act as attractors that, if the system is perturbed, can pull the system towards that steady state. Gottman et al. (2002) utilized a metaphor of a rubber band to describe the properties of attractors; if the band is stretched, it snaps back into its original state when it is released. Systems with stable steady states work in a similar fashion.

Gottman et al. (1998) used equations that yielded scores based on two components: (1) The interpersonal influence from one spouse to the other, and (2) the individual's own dynamics (or the uninfluenced behavior). The uninfluenced behavior may consist of their present emotional state, any past interactions that the couple have had (particularly with regard to conflict), any prior experiences (e.g., family-of-origin experiences), or any personal characteristics and dispositions (e.g., dysphoria or optimism). The influence component was defined as function of one person's ability to move or change the other person's emotional state (positively or negatively) from the uninfluenced steady state. Further, Gottman and his associates were able to model the process for repairing and dampening the other person's emotional states, which created more steady states. Ideally, the more and better positive steady states, and the fewer negative steady states that a relationship has, the more satisfied and regulated the marital system is. This modeling has allowed for predictions about changes in the relationship to be made, and how to best intervene with the couple.[1]

This work has not been without critique, however (see Stanley, Bradbury, & Markman, 2000, and Gottman, Carrère, Swanson, & Coan, 2000, for an in-depth critique and response). In particular, Heyman and Slep (2001) have cautioned against the overgeneralization of prediction without additional cross-validation (a point that Gottman, Swanson, & Swanson, 2002 and Coan & Gottman, 2007, subsequently addressed). Vallacher et al. (2010) specifically affirmed Gottman and his colleagues' approach as a viable application of dynamical systems to real-world relationships (or "conflicts").

Application to Therapeutic Treatment: Formulation of the Mathematical Model

While the original target of Gottman's work was married couples, his use of dynamical systems and the equations used to model the interaction can be customized to reflect the unique characteristics of the therapist-client relationship. Of particular interest is the emotional influence of the therapist and client on each other in the creation of an effective (or ineffective) therapeutic relationship. Like couples, we would expect that the theoretical mathematical model could be used to describe and predict successful and unsuccessful (therapeutic) relationships depending on the parameters or conditions of the relationship. It is important to note that we are not reproducing the dynamics of marital relationships and applying them to the therapeutic relationship. Instead, we are using a dynamical systems approach (which has been used to model various other dyadic interactions, such as couples' relationships) and applying the unique properties of the therapist and client relationship. Thus, modeling the therapeutic relationship between therapists and clients should allow researchers to be able to evaluate the quality of the relationship and the effectiveness of specific interventions that might create some significant therapeutic gains with a predictability that has not yet been seen in the clinical research literature. The information would allow researchers to study this relationship in detail, to be able to ascertain the parameters of the relationship, and to be able to see how specific intervention strategies can *predict* changes in clients as well as see how specific intervention strategies actually *produce* changes in clients.

The starting point for the present project is the dynamic nonlinear mathematical equations that Gottman et al. (2002) used to model the emotional valence of marital dyads. We alter these equations to uniquely model the elements of the therapist-client relationship. We will also demonstrate—through computer simulations of the solutions of these equations—how varying some of the parameters creates new dynamics within the therapeutic relationship, and how this might affect the outcome of

therapy. Lastly, we will discuss some insights for clinicians, and future directions for further assessing the therapeutic relationship and validating the various models found in the simulations.

The Equations of the Therapeutic Relationship

In the model, we use the differential equations that Liebovitch et al. (2008) derived from Gottman et al.'s (2002) original marital difference equations (see Appendix for details). However, the major difference is the creation of influence functions for both the therapist and the client. These influence functions are essentially the mathematical "blueprint" for how each "actor" in the system will behave in reaction to the other "actor" (i.e., how the therapist responds to the client or how the client reacts to the therapist). The influence functions were created by considering what would be a "typical" therapist's emotional reaction to a client's display of negative, neutral, or positive emotion. We likewise considered what a client's reaction would be to a therapist's display of negative, neutral, or positive emotion (see below for details). These influence functions were based on both our own clinical experience as well as from the literature on the therapeutic alliance (Bohart & Tallman, 2010; Duncan, Miller, Wampold, & Hubble, 2010; Gelso, 2009; Gelso & Hayes, 2002; Horvath & Bedi, 2002; Horvath et al., 2011; Lambert & Barley, 2002; Norcross, 2002, Norcross, 2010; Norcross & Wampold, 2011; Safran, Muran, & Eubanks-Carter, 2011; Safran, Muran, Samstang, & Stevens, 2002) and the literature on the behavior of expert therapists (Skovholt & Jennings, 2004). The result is two distinct influence functions that create a picture of what each person in the therapist-client dyad would be doing in relation to the other person at different levels of affect (from extremely negative to extremely positive). It is important to note that, at this point in the project, these influence functions are *speculative* in nature and provide a starting point for this exploratory project. They are presented visually in Figure 1(a–c).

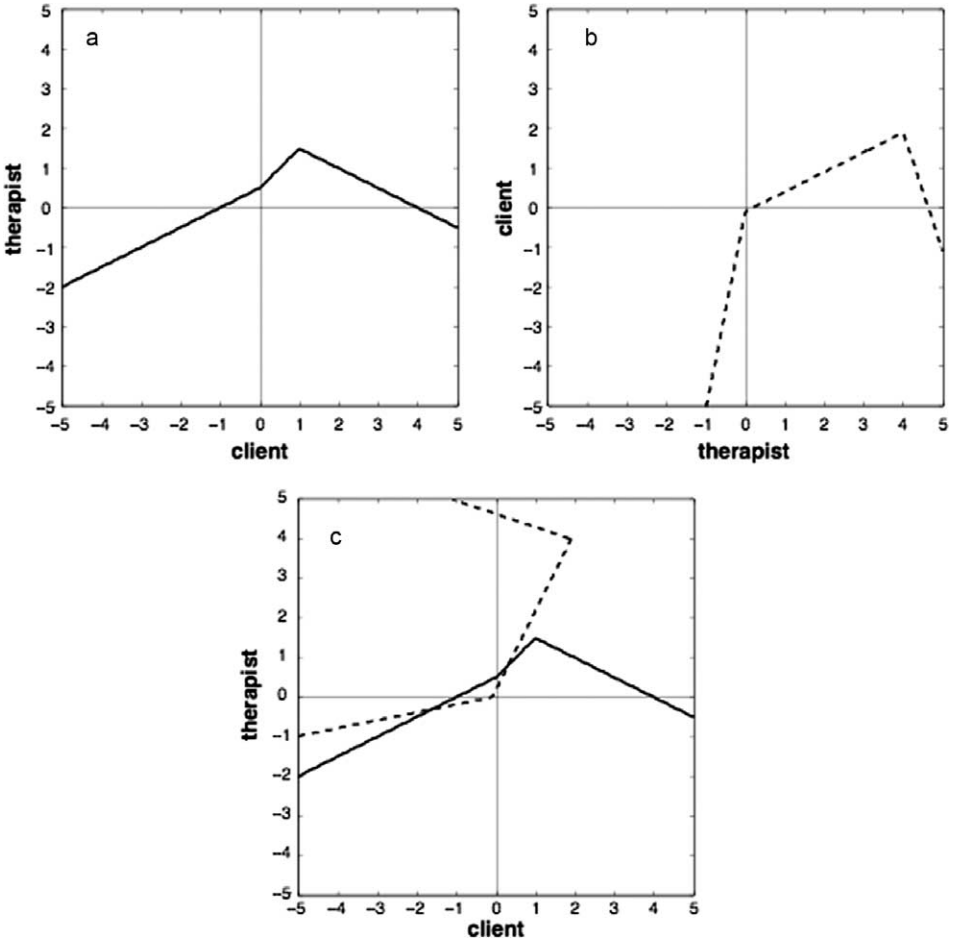

Figure 1. The influence functions for the model where thetherapist and client respond with equal intensity to each other. How the client influences the therapist is shown in (a) and how the therapist influences the client is shown in (b). The intersection of these functions, shown in (c), identifies two critical points (in the upper right and lower left).

Description of client's influence function. Figure 1(a) shows the client's influence function on the therapist's behavior.

Negative affect. When the client's affect is negative, the therapist will most likely exhibit more positive affect to try to draw out or encourage the client. However, they may, under prolonged "exposure" to the client's negative affect, begin to exhibit neutral and even negative affect in the face of extreme negative behavior. This would create a steady state in the negative-negative space, which would effectively be the death of therapy—a "black hole" from which the therapeutic relationship dies (Bohart & Tallman, 2010; Gelso, 2009; Gelso & Hayes, 2002; Horvath & Bedi, 2002; Horvath et al., 2011; Norcross, 2010; Norcross & Wampold, 2011; Safran et al., 2002, Safran et al., 2011).

Neutral affect. When the client is affectively neutral, therapists will generally utilize strategies to elicit more positive emotions. They will attempt to encourage clients, or try to get the client to focus on their strengths and abilities, in the hopes that this change of focus will change the client's affect (Bohart & Tallman, 2010). At the same time, therapists may try to provoke any affect from the client (which may sometimes be negative). However, unless tied to a broader strategy, this is generally born out of frustration and may undermine the therapeutic alliance (Gelso, 2009; Horvath & Bedi, 2002; Lambert & Barley, 2002; Norcross, 2010; Safran, et al., 2002, Safran, et al., 2011). We will discuss this below.

Positive affect. As the client's affect moves from neutral to positive, initially, the therapist will also exhibit more positive affect. However, there is a point where, as the client's affect becomes more positive, the therapist may begin to take a more neutral affective stance, as the therapist no longer needs to actively "encourage" the client, but the positive affect (presumably from some positive behavior change or symptom relief) sustains itself (Bohart & Tallman, 2010; Gelso, 2009; Gelso & Hayes, 2002; Horvath & Bedi, 2002; Lambert & Barley, 2002; Norcross, 2002; Norcross & Wampold, 2011; Safran et al., 2002).

Description of therapist's influence function. Figure 1(b) shows the therapist's influence function on the client's behavior.

Negative affect. When a therapist exhibits negative affect, the client is likely to react even more negatively. The client may experience therapist negative emotion as judgmental, or a signal of some disappointment in the client. This may be the result of therapist frustration with either the pace of treatment or the client's unwillingness to change, or fears about the therapist's own performance in conducting therapy (Anderson, Lunnen, & Ogles, 2010; Bohart & Tallman, 2010). It is reasonable to suspect that this would be a part of novice therapists' practice, but could also be reflective of therapists who may be on the brink of burnout. The therapist may not even acknowledge the frustration, but it may get picked up by the client (Safran et al., 2011), and move the therapy towards the more negative end of the graph. This is an indicator of a therapeutic rupture, which in turn is a predictor of premature termination form therapy (Muran et al., 2009; Norcross, 2010). At the same time, there may be circumstances when a display of negative emotion may be beneficial to the therapeutic relationship. In particular, appropriate confrontation or expressions of disappointment may be necessary feedback to the client (Safran et al., 2011). Again, the immediate result may be a therapeutic rupture, but if it is done purposefully or strategically, it may have a long-term benefit for the client. The success of this strategy depends a lot on the skill of the therapist and the strength of the therapeutic relationship (Anderson et al., 2010; Gelso, 2009; Gelso & Hayes, 2002; Horvath & Bedi, 2002; Lambert & Barley, 2002; Norcross, 2002; Norcross & Wampold, 2011; Safran et al., 2002, 2011).

Neutral affect. When the therapist is affectively neutral, most clients are likely to be either slightly negative or neutral (particularly early in the therapeutic process). Some clients may not be influenced one way or another to a therapist's neutral affect, unless they find (i.e., project) it to be a signal of therapist disinterest (e.g., the "tabula rasa" of psychoanalysis), at which point clients may react negatively, feeling that the therapist is too detached or "clinical" (Bohart & Tallman, 2010; Gelso, 2009; Gelso & Hayes, 2002; Horvath & Bedi, 2002; Lambert & Barley, 2002; Norcross, 2010; Safran et al., 2002).

Positive affect. As the therapist's affect moves from neutral to positive, initially, the client may remain neutral, or slightly negative (Bohart & Tallman, 2010; Gelso, 2009; Gelso & Hayes, 2002; Horvath & Bedi, 2002; Lambert & Barley, 2002; Norcross, 2002; Safran et al., 2002). However, as the therapist's affect becomes more positive, the client may respond positively by exhibiting more neutral affect (Skovholt & Jennings, 2004; Safran et al., 2002). This could be a sign of the client either "buying into" the therapist's message, or a sign that the client is beginning to experience some positive

results from the therapeutic intervention. A positive steady state may emerge at this point, where therapeutic gains may be maximized (Norcross, 2010; Norcross & Wampold, 2011). However, as the extreme expressions of positive affect on the part of the therapist, the client might turn "negative" (i.e., get "turned off", especially if they perceive that it is disingenuous or too "pollyannish") (Horvath et al. 2011; Safran et al., 2011).

Analysis of the Mathematical Models

We now analyze the dynamics of the mathematical model by: (1) First identifying the values of the behavior state of the therapist and client that define the "critical points"[2] which can represent the final steady-state values that they reach at the conclusion of therapy, (2) then determining which values of the initial behavior state of the therapist and client will reach those final steady states, and (3) finally investigating—through computer simulation—how the dynamics of the relationship between therapist and client depends on the parameters of the model, and how the relationship changes based on the initial parameters of the models (i.e., different therapist-client combinations).

First, it was important that we plot both the client and therapist influence functions on the same graph. The points where they intersect tell us the critical points that could represent their steady states or attractors for the system. This is illustrated in Figure 1(c) where the two influence functions are plotted on a graph of the behavior state of the therapist versus the behavior state of the client. On that plot we can see two intersections that identify two critical points.

The stable critical point in the upper right of Figure 1(c) is a good outcome, because both the therapist and client have positive emotional states for that stable steady state. It is at a point where the therapist has more positive affect than the client. This seems—for this simulated system—to be the most beneficial spot for a therapeutic relationship to be in. One interpretation is that, at this point, the client is at least in the positive space, but may be trying to consolidate the new behaviors or way of thinking with their present life. If they have been initially successful, then the therapist will have a good working relationship with the client to allow him or her to generalize to other areas in the client's life (Norcross, 2010; Norcross & Wanpold, 2011).

The unstable critical point in the lower right of Figure 1(c) is in a negative-negative space. We will see (below) that the dynamics near this critical point drags the client into ever more negative behavior states. Therapeutic relationships that move in this direction are in serious jeopardy of becoming entrenched in a fruitless struggle, develop a therapeutic rupture, and eventually lead to a termination of the relationship (Muran et al., 2009; Safran et al., 2011). At this point, the therapist, despite previous efforts at using positive or neutral affect to elicit some positive behavior change for a client, may—out of desperation or frustration—resort to expressing negative emotion to provoke a reaction. However, this is often done without much reflection and is frequently ineffective (Anderson et al., 2010).

Dynamics of the Therapeutic Relationship

We now continue the analysis of the model presented in Figure 1 by determining the dynamics, how the expression of emotions between the therapist and client evolve together in time. We do this by using a plot called a "phase-space."[3] Each point in this phase space corresponds to one value T for the behavior state of the therapist on the vertical axis and one value C for the behavior state of the client on the horizontal axis. The therapist and client each start with an initial behavior state called their "initial condition." Their initial condition, having one value of T for the therapist and one value for C for the client, is represented by one point in the plot. The dynamics of the therapeutic relationship can then be visualized by following the path of this point, called its "trajectory," as it moves through the plot (much like a swimmer caught in a current) towards or way from an attractor.

Figure 2 shows the phase-space plot of the trajectories of the therapist and client where the therapist and client have equal influence (meaning that the client and the therapist react emotionally to each other in an equal way) with each other. This corresponds to the same parameters and influence functions shown in Figure 1(c)[4]. The lines (trajectories) in Figure 2 clearly show that many initial conditions are attracted to a stable endpoint at the critical point in the upper right corner or they will be attracted to the critical point in the lower left of the figure. For example, if a therapist starts the therapeutic interaction at a 5 and the client starts at a $+3$, the point were the two intersect is the initial condition. The line that passes through that point represents the trajectory that the relationship is projected to take toward the stable point. This does not mean that the relationship is predetermined to go to that point, but that, according to the system dynamics, it indicates that—over time—unless there is an effort made to move away from the attractor point, the relationship will end up at the homeostasis point emotionally.

In this instance, where client and therapist have equal influence with each other, the relationship will

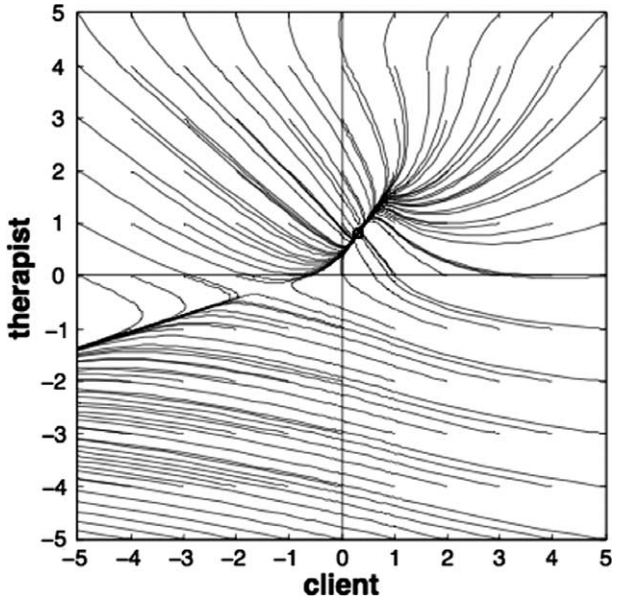

Figure 2. Phase-space plot of the dynamics of the therapeutic relationship corresponding to equations (1a) and (1b) and the influence functions in Figure 1. In this model the therapist and client respond with equal intensity to each other. Each point in the plot reflects the behavior state of the therapist (vertical axis) and the client (horizontal axis). At the stable steady-state attractor both the therapist and client are in mildly positive states.

likely end up in at this positive attractor, as long as the therapist begins with positive emotion. However, if the client starts therapy in a very negative emotional state (where "*C*" starts at 5 or 4) then the therapist must be more positive in order to overcome the movement towards the negative state ("*T*" starting at +2 or higher). Furthermore, in this model, if client begins therapy with a mild negative state (−1) or is neutral, the therapist can also match the negative emotion and still attract the relationship towards the positive stable steady state (approximately +1 for therapist and client). In addition, if the client starts therapy with very positive affect (+2, to +5) the therapist can also display some negative or neutral emotion and still draw the relationship to the positive steady state. "Going negative" can be a strategy for the therapist to either bring a client who is mildly negative or neutral (−1 or 0) about therapy into a positive space. It may also be a strategic method for "tamping down" a client who is displaying highly (and possibly unrealistically) positive emotions. Since, in this scenario, both the client and the therapist are equally influential of the other, one can look at the other side of the coin. Specifically, if the therapist initially is highly positive (+5 to +2) and the client is either negative or positive, the therapist will be drawn down toward the positive stable steady state. The key, it seems, for this relationship is that (in most instances) the therapist must avoid beginning with a negative affect (Norcross & Wampold, 2011). The only exception is if the client is initially very negative, in which case, the relationship will invariably be pulled towards negative emotional states.

Producing different models by changing the parameters of the therapeutic relationship. The dynamics of the therapeutic relationship shown in Figure 2 represents our first model using our basic set of parameters and influence functions. The power of a mathematical model is that we can now explore, through computer simulations, what happens when we change those parameters. In particular, we varied the parameters of the therapist's or client's reactions to each other's influence, and the strength of therapist's or client's response to their previous emotional state (or level of inertia). In other words, how reactive one person is to the other person's emotional display determines the level of influence. In addition, how strong a person's response is to their previous emotional state is a reflection of the level of inertia or entrenchment that one person has to how they have been feeling all along, as well as how quickly they may be willing to abandon or consciously change it. In each of the figures that follow, one or more of the parameters of the equations (1a) and (1b) has been changed from the base model in Figure 2 to simulate different aspects of the relationship between therapists and clients. We discovered that there were three distinct "types" of relationships that emerged (in addition to the initial model detailed above), which will describe each below.

Model 1: The highly responsive client (or influential therapist) yields positive outcome. Figure 3 shows the phase-space plots of the trajectories of the therapist and client if the client responds very strongly to the therapist ($s_{T\rightarrow C} = 10$ rather than 1). This may be in indication of a very influential or skilled therapist. Just as in Figure 2, there are two critical points, one of which is a stable steady-state attractor where the client is very positive and the therapist is moderately positive, and the other is an unstable critical point. There are some noteworthy issues. First is that the therapeutic relationship is attracted toward the positive steady state. Second, the emotional state of the relationship spirals, or oscillates up and down in time before reaching a final steady state. This seems to be in line with clients' frequent oscillations (i.e., ambivalence) regarding change. Third is that the client winds up more positive than the therapist, and that the relationship is attracted to the steady state rapidly (as indicated by the "tight" spiraling). This seems to be a very good outcome for therapy. Fourth is a cautionary finding that the client will likely respond very strongly to any

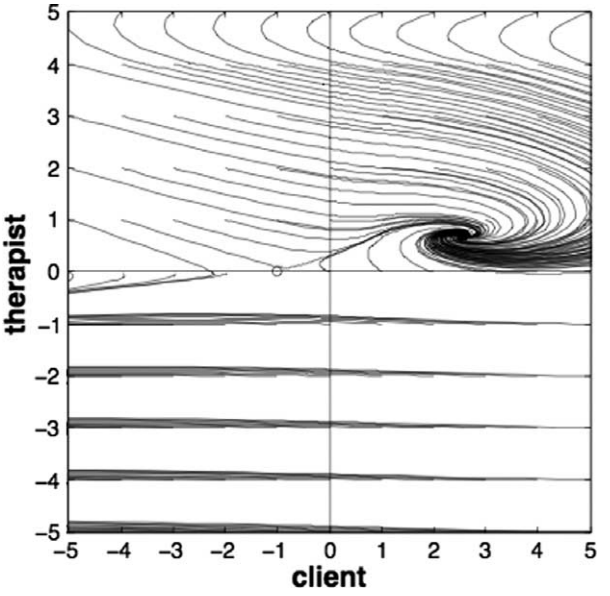

Figure 3. Phase-space plot of the dynamics of the therapeutic relationship when the client responds very strongly to the therapist. There is one stable steady-state attractor (middle right) where the client is in a positive state and the therapist is in a mildly positive state. The other trajectories are drawn towards the increasingly negative states (lower left).

negative input from the therapist. As shown in Figure 3, any time that the therapist starts with a negative "score" on the therapist axis, the relationship will be directed towards the unstable critical point and from there to increasingly negative values. On the other hand when the therapist starts with positive scores, the relationship will be directed towards the positive steady state. The only exception to this is when the client scores start very negative (-5 to -3) and the initial therapist scores are only neutral or slightly positive therapist scores (0 to $+1$). Thus, just as in Figure 2, as long as the therapist begins with positive emotion, the client will reach this positive outcome, unless the client starts very negative about treatment. Fifth, this model may be an "ideal" scenario for a brief therapy, where change is swift (as indicated by the close lines) and the client is satisfied. Second, just as in Figure 2, as long as the therapist begins with positive emotion, the client will reach this positive outcome, unless the client starts very negative about treatment.

An additional simulation (not illustrated) was run that modeled a relationship if the client responds more weakly to their own previous state ($a_C = -0.1$ rather than -1.0), but with equal influence ($s_{T\to C}$ and $s_{C\to T} = 1$). This may be indicative of a client who is very motivated to change (i.e., responds weakly to their pervious state) because they are not so "entrenched." The client may not be so rigid in his or her own point of view, or they are so "sick and tired" of how bad things have gotten that they are not clinging to their old ideas or behaviors. As a result, they are open to accepting influence from their therapist. This is a very positive or optimistic therapeutic relationship. As with Figure 3, there are two critical points, one stable and one unstable, both in the same relative places as in Figure 3 (as a result, we have omitted these phase-space plots), though there are several important points worth discussing. First, if the client begins with very negative emotion, the therapeutic relationship is unlikely to move towards the positive attractor, no matter how positive the therapist is initially. Clearly, in this scenario, if the client is very negative (e.g., toward therapy—a "precontemplator"), then the therapist taking a neutral stance is probably not going to effect much change. Second, the movement here towards the steady states is slower, therefore the therapist needs to be patient with the pace of therapy, otherwise the client may pick up on the therapist's frustrations and the relationship become attracted to the negative steady state. This is important feedback to therapists in training, or therapists who may be struggling with clients who are ready for change, but might not have much hope that they can be helped (i.e., feeling very negative and "sick and tired" of the status quo).

Model 2: Being too responsive to clients (and not very influential) produces mediocre outcomes. Figure 4 shows the phase-space plot of the trajectories of the therapist and client if the therapist responds very strongly to the client

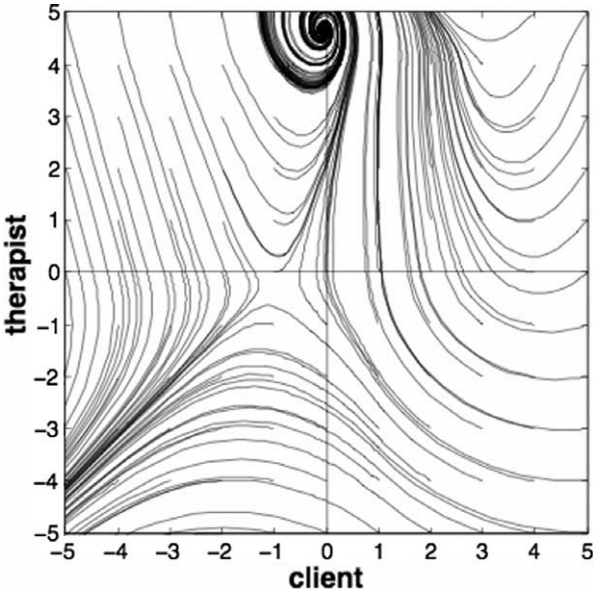

Figure 4. Phase-space plot of the dynamics of the therapeutic relationship when the therapist responds very strongly to the client. There is one stable steady-state attractor (middle top) where the therapist is in a positive state and the client is in a neutral state. The other trajectories are drawn towards the increasingly negative states (lower left).

($s_{C\rightarrow T} = 10$ rather than 1). Again, there are two critical points, one stable and one unstable. However, in this case, the stable attractor is located where the therapist is very positive and the client is neutral. Again, there are oscillations between positive and negative feelings before settling on the steady state. Looking at Figure 4, and following along, the client is very influential and persuasive, or the therapist is very responsive to the client. In this model, if the client starts negatively (from -5 to -3), no matter how positively the therapist begins the therapy, the relationship will be drawn towards the unstable critical point and towards ever more negative states (a very poor outcome). If the client starts mildly negative (-1 to -2), and the therapist starts neutral or mildly positive, the relationship will also be drawn towards these negative outcomes. If the therapist is more positive when the client starts slightly negative, then the relationship will be attracted towards the other steady state where the therapist is positive and the client is neutral. However, if the client starts positive ($+1$ to $+5$), the therapist can be either positive or negative, and the relationship will be attracted to the steady state. For most clients, this will mean moving from a positive emotional state to a neutral state (and probably not a great outcome). However, there may be some scenarios where this is a "positive result" therapeutically. For example, if a client is unrealistically positive (e.g., euphoric or manic), then employing strategies (even using/displaying negative emotion) to bring them towards a more realistic view (e.g., neutral) might be beneficial. In addition, clients who are mildly negative initially, in this scenario, may reach a neutral state (a better outcome emotionally than at the beginning). In most cases, however, this configuration is not optimal for successful therapy.

Another simulation (not illustrated) was run with the parameters of the influence functions set for the therapist and client if the **therapist responds more weakly to their previous state** ($a_T = -0.1$ rather than -1.0, and $s_{T\rightarrow C}$ and $s_{C\rightarrow T} = 1$). Again, because of the similarity of the plot, it is not re-printed here. In this phase-space plot, clients who begin therapy with negative emotion can very quickly move to a positive feeling state and then to neutral, as long as the therapist is neutral or positive, emotionally. The only appreciable difference to Figure 4 is that, since the therapist responds more slowly to changes, it takes more time for the therapeutic relationship to approach the steady state (again, because the lines are spread further apart). This may represent a therapist who is more patient with the client, especially clients who are initially negative about therapy, but who are "won over" (at least in part). The problem is that, in this model, there is a strong pull towards the negative attractor that may be tough to overcome. Another explanation for why this may be happening is that novice therapists in this situation may seem "wishy-washy" or unsure about their strategy or approach to therapy. Therefore, they may seem to be less directive, which can become frustrating to the client who is seeking direction or more active help from the therapist. The therapist may feel good about the therapy they are delivering, and think that they are being attentive to the client, but the client probably doesn't feel very satisfied. The client may feel like the one who is "leading" the therapy and that it is not "going" anywhere (or moving forward), or that therapist is not doing much. The key for the best outcome in this model is a positive emotional stance by the therapist.

Model 3: Client and therapist exert strong influence and produces complex system. Figure 5 shows the phase-space plot of the trajectories of the therapist and client if both the therapist and the client respond very strongly to one another ($s_{C\rightarrow T} = s_{T\rightarrow C} = 10$ rather than 1). There are a total of four critical points. Two are stable attractors and two are unstable critical points. The stable attractors that are present roughly correspond to the attractors in Figure 3 (where the client is more positive than therapist) and Figure 4 (where the client is neutral and the therapist is positive). In addition, there is unstable critical point where the therapist is neutral and the client is slightly negative and another unstable critical point where both the therapist and

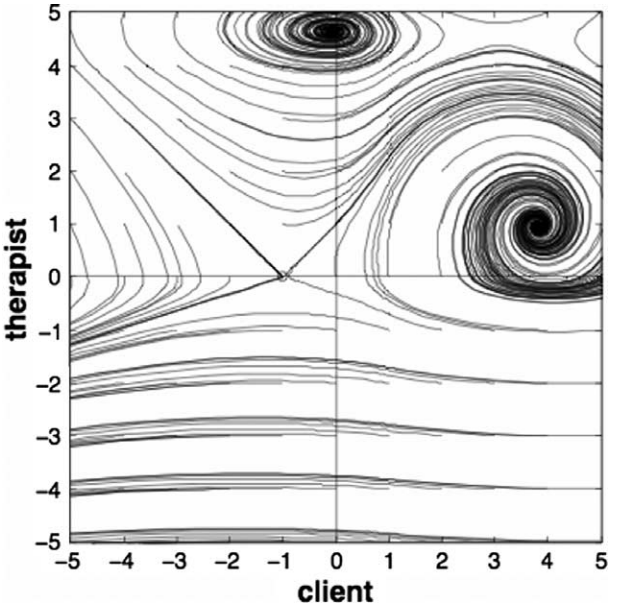

Figure 5. Phase-space plot of the dynamics of the therapeutic relationship when the therapist and client both respond very strongly to each other. There are two stable steady-state attractors, similar to those in Figures 3 and 4.

client are positive. This is one of the more active and dynamic systems for therapy, as there are three potential outcomes. First, if the client is initially extremely negative (-5 to -2), and the therapist is initially neutral, or not positive enough ($+1$ or $+2$), the relationship will be attracted towards the very negative region. Second, if the therapist is initially extremely positive, and the client begins slightly negative, neutral or slightly positive, then they will end up at the neutral attractor (for the client). Third, however, if the therapist is initially positive and the client is initially negative, or if the therapist is initially positive to neutral while the client is initially positive, then the relationship will end up at the attractor which is positive for both. In other words, therapist neutrality (unless in the face of very positive emotion from the client) is not always predictive of good outcome for the relationship. Additionally, the client will either be "won over" to the far positive steady state, or will stay entrenched in a neutral affect (and probably terminate).

Therapists who react very strongly to their clients may seem to be really "in tune" with their client. They may be able to connect with their clients quickly, and be able to intervene effectively with their clients. However, they may also be more sensitive and either over-react or react negatively to the client. At the same time, clients who react strongly are apt to be influenced by their therapists. These clients may not necessarily be motivated for change initially (i.e., if they initially begin therapy in a negative state), but may "feed off" the therapist's enthusiasm and influence. This may be indicative of clients who are emotional labile and prone to shifts in mood.

As with Figures 3 and 4, an additional simulation was run (not illustrated) where the parameters for the influence functions were changes so that both the therapist and the client respond very weakly to their previous state ($a_C = a_T =$ if, 0.1 rather than if, 1.0). Again, the result is very similar to Figure 5 (with two stable attractors and two unstable critical points) except that more time passes before the relationship arrives at a stable steady state (because the spirals are not a close as in the other figures). This type of therapeutic relationship would have therapists and clients alike who may be more present-focused, and more process-oriented. They may be more reflective and introspective, but may also be more likely to adopt a solution-focused approach where the therapist and client collaborate on solutions rather than problems (see Duncan, Miller, Wampold & Hubble, 2010). Thus we would expect many of the same dynamics to result as the model in Figure 5, but taking longer to achieve.

Conclusion

The purpose of this paper was to show that dynamical systems could be used to model the emotional exchange and strength of the therapeutic relationship in a way that would provide useful information for clinicians. The simulations were run on the basis of the influence functions, and yielded some interesting theoretical models of what happens in effective and ineffective therapy (i.e., symptom reduction, problem resolution, and client satisfaction). From the simulations, four different models emerged so that, after investigating the different phase-space plots, we can begin to argue or hypothesize that some strategies (or approaches) will be more effective given certain conditions or parameters. While it can be said that this is already being done in the literature (e.g., Gelso, 2009; Muran et al., 2009; Safran et al., 2011), these simulations give us a starting point for both explaining and predicting why this happens in therapy. Gottman et al. (2002) seemed to understand the potential of this approach beyond the marital dyad:

> The model (that was developed) has given birth to a *theoretical language about the mechanism of change* - . . . The model provides the language . . . suggests variables that can be targeted for change using interventions. In short, the model leads somewhere. It helps us raise questions, helps us wonder what the parameters may be related to and why. It raises questions of etiology . . . Thus, it is likely that the major contribution of the model will be the theoretical language and mathematical tools it provides. It will give us a way of thinking . . . that we never had before. (2002, p. 172, parentheses added)

With these simulations, clinical researchers can begin to predict the strength or quality of the therapeutic relationship (and subsequently outcome in therapy) based on the value of certain initial values at the beginning of therapy. Using dynamical systems we can make predictions about how negative outcomes result from some of these beginning values, and (more importantly) how to switch strategies (or approaches) in therapy to get a better outcome.

Implications for Practice

There were several interesting findings that the simulations revealed, which are worth pursuing. First, an overall analysis of the models shows that the person (therapist or client) who is most responsive to the other winds up being the most positive, that is, in the positive steady state. This seems, on its

face, to support Gelso's (2009) assertion that therapist neutrality is not always the best strategy, and may even be detrimental to the therapeutic relationship (as indicated by Norcross & Wampold, 2011). Next, in most, but not all, of these examples, the steady state is reached through a spiral trajectory. That is, the client (and therapist) will go through up and down emotional swings before reaching their final steady states. This seems to be validated by clinical observations dating back to Freud and his concept of resistance, regarding clients vacillating and displaying ambivalence about making lasting changes. This is also reflective of the work of Miller and Rollnick (2002) and the Motivational Interviewing approach on resolving ambivalence rather than merely "confronting" it (specifically in light of the finding that the therapeutic relationship suffers when therapists are negative). Third, for most simulations, the client winds up in a very negative emotional state if the therapist starts in a negative emotional state. This was an overwhelming finding, and—though it might seem self-evident—it should be reinforced to practitioners and trainees everywhere. Fourth, for most simulations, even if the client starts in a negative emotional state, if the therapist is initially neutral or positive, the client will wind up in a positive emotional state. The impact of the valance of the relationship (positive or negative) is crucial in the ultimate outcome of therapy:

> The client may genuinely and realistically not especially like the person of the therapist, or the client may realistically perceive or "subceive" negative reactions on the therapist's part of which the therapist is perhaps unaware. There can indeed be negative reactions within the context of a real relationship. At the same time, negative valence would reflect a weaker real relationship. (Gelso, 2009, p. 255)

By and large, this is an optimistic finding for the practice of psychotherapy, and also seems to support over 50 years of findings that psychotherapy is generally effective (Lambert & Barley, 2002; Norcross, 2010; Norcross & Wampold, 2011).

Also, we found that increasing the influence of the other person yields the same type of phase-space trajectories as responding more weakly to the previous emotional state. The difference is that responding more weakly to the previous emotional state—i.e., increased resistance to change—means that it will take a longer time to effect change. Thus, the two key variables that therapists must be mindful of is the amount of influence they and the client have over each other in a given therapeutic relationship, and how strong the response is to their own previous emotional state. On the therapist's part, this "resistance" might be indicative of long-held beliefs about therapy (theoretical approaches, stance of the therapist, etc.), and for the client, it might be the result of their own long-held beliefs about life or whether they are ready for (or feel that they need to) change. However, in light of Gelso's (2009) finding about the role of the "real relationship" in the therapeutic endeavor, understanding how these variables affect the emotional valence of the relationship can have a direct effect on the strength of the therapeutic relationship and working alliance. Indeed, as Hill and Knox (2009) have detailed, monitoring and addressing these relationship issues are important in enhancing outcomes. The present findings can provide a roadmap for clinicians to monitor themselves, their clients, and the therapeutic relationship, relative to the emotional valence of both the client and the therapist. In addition, therapists can determine which of the four models their particular therapist-client relationship is following, and whether they are heading towards a positive or negative steady state. At the same time, these findings must be evaluated in a real-world context with actual therapeutic relationships. This is the next step in this work.

Future Research

While the results of this investigation are both exciting and promising, there are several questions that remain to be answered. Thus, the next logical step in this project is to subject the models that emerged from the simulations to empirical testing. Interestingly, Aharonovich, Amrhein, Bisaga, Nunes, E. & Hasin (2008) recommend to researchers that the "dynamic aspect of the (therapeutic) process can be studied empirically through analysis of codes assigned to patient statements in recorded therapy sessions" (p. 557, parentheses added). As Gelso (2009) noted, the therapeutic relationship is both a "real" and a professional relationship where the therapist is explicitly trying to influence the client, and the client will likely be resisting this influence (actively or passively). Therefore, we would expect that the dynamics of the therapeutic dyad would reflect a more direct and active process of influence by the therapist as well as active influence (resistance or reactance) by the client. Following from the simulations of this relationship, we can create some testable hypotheses about how to tailor therapeutic interactions based on the parameters of these models. What we don't know is how long each of the models identified takes to get to the steady state. We also don't know what exactly makes up very negative, negative, neutral, positive, and very positive affect.

Specifically, therapists and clients could be videotaped and their therapeutic interactions could be coded to see if they fit one of the four models discovered through the simulations in the present paper (or if new models may need to be created to better fit with the observed data). This is one of the strengths of a dynamical systems approach (Liebovitch et al., 2008; Vallacher et al., 2010). The initial session could be videotaped and coded, and then a follow-up session could be recorded to see the fully developed therapeutic relationship. In addition, research with real therapist-client dyads would be able to designate those relationships that make up successful therapy, and those that do not, based on clinical outcome measures, client satisfaction measures, and self-report measures of the therapeutic relationship. It would be possible to sample both expert and novice therapists to see if there are any systematic differences between them regarding the models that they fit into. Lastly, it might be possible to discover how to intervene strategically with a client or with a therapist when the relationship is pulled toward a negative attractor. Another powerful aspect of this kind of research would be identifying how the affective elements of the therapeutic relationship impact the mechanisms of change over time (either positively or negatively). At this point, specifically identifying the amount of time as a parameter is one of a number of empirical questions that we are interested in pursuing, but that we cannot definitively answer at this time (see Liebovitch, Peluso, Norman, Su, & Gottman, in press, for more details).

A frequent problem that researchers using dynamical systems have faced is estimating the parameters from the data and then applying the best fit to the data. Models can be created, but if there is no way to evaluate their goodness of fit to real data, then the models are illustrative and speculative at best, but may not hold up to the most basic principle of science: falsifiability. The parameters of the model determine how the emotional state of the therapist and client depend on time. This predicted time series can be compared to an experimentally measured time series developed by a suitable coding of the emotional states measured from video recordings. The parameters can then be found, through numerical methods, that minimize the difference in the sum of the squares of the errors between the predicted and measured time series. This was the method used by Gottman et al. (2002). Recently, Hamaker and her colleagues (Hamaker, Zhang & Van der Maas, 2009) also devised a method for estimating the parameters for a model and compute a Bayesian Information Criterion (BIC) which can then be used to compare different influence functions and determine the best fit for the given interaction. In fact Madhyastha, Hamaker, and Gottman (2011) have recently used this to test Gottman et al.'s (2002) original influence functions with interesting results. Thus, employing this same approach, it might be able to compare the different influence functions for therapists and clients and determine the best-fitting dynamical systems model, which may solve the problem of determining the effect of therapist "responsiveness" (differing levels of affective involvement that emerges within the context of client needs) on client outcomes that Norcross and Wampold (2011) raised.

Summary

The purpose of this paper was to demonstrate the usefulness of modeling the affective nature of the therapist-client relationship using dynamic systems. Based on the influence functions created to describe the client's and therapist's reactions, we have developed four archetypal therapeutic relationships based on this computer simulation of the parameters (see Appendix). These theoretical models provide a starting point for a larger discussion for both clinicians and researchers regarding the dynamics of the therapeutic relationship. In addition, there may be more "types", based on changes to the influence functions. Research using actual therapist-client dyads and coding the affective exchange is the next step in empirically validating, or altering, these models. While it may not be realistic to expect clinicians to routinely videotape and code the interactions with clients, we hope that future research might be able to better define the ranges of negative, neutral and positive affect, and provide clinicians with some clear exemplars of each within a therapeutic relationship. The ultimate goal is to detail the elements of the therapeutic relationship that can reduce the rates of premature dropout, and increase the effectiveness of clinicians. By looking at the mechanisms of the therapeutic relationship, in depth and detail, we aspire to shed some light into this important area.

Notes

[1] Once the equations were completed, Gottman and his associates sought to test their models on couples. Couples were videotaped, and the quality of their interaction was coded using the Specific Affect Coding System (SPAFF; Gottman et al, 1996). The SPAFF codes specific emotional behavior of a husband and wife in real time, and can be used in any conversation. According to Gottman et al., the SPAFF focuses solely on the affects expressed by the participants, drawing on facial expressions, vocal tones, and speech content to characterize the emotions that are displayed. SPAFF coders "categorized the affects displayed using five positive codes (interest validation,

affection, humor and joy), 10 negative affect codes (disgust, contempt, belligerence, domineering, anger, fear/tension, defensiveness, whining, sadness, stonewalling), and a neutral affect code" (2002, p. 179). The final weighted scale ranged from 24 to +24, giving equal weighting to the five positive codes and the 10 negative codes. Couples were asked to have a 15-minute discussion about an area of ongoing conflict, which was videotaped for coding using the SPAFF. Each videotape was coded in its entirety by two independent observers, and then used to determine the parameters of the previously formulated nonlinear equations of the mathematical model. The results of their analyses yielded support for the predictive ability of the parameters of the model to discriminate between three separate criterion groups (happy stable couples, unhappy stable couples, and divorced couples), as well as discover the importance of each of the four parameters (husband and wife's influence and uninfluenced steady states) and the ability to intervene therapeutically (for a full discussion of these results, please see Gottman et al., 2002).

[2] Critical points can be "stable" or "unstable." If small changes in T and C bring both the therapist and client back to their values at the critical point, then that critical point is a stable steady state. This is a point of equilibrium that the relationship would gravitate towards (or be attracted to). It is therefore called an "attractor" of the relationship. However, if small changes in T and C always push the therapist and client further away from the critical point, then it is unstable and does not represent a final steady state. We will see in the computer simulations below that the critical point at the intersection of the influence functions in the upper right of Figure 1(c) is a stable steady state, while the critical point at the intersection of the influence functions in the lower left of Figure 1(c) is unstable. This has important (and potentially useful) implications for the therapeutic relationship.

[3] We used computer software (MATLAB ODE113) to calculate how the values of T and C, for the behavior state of the therapist and client respectively, in equations (1a) and (1b), change in time when they start from many different initial conditions. We then plot these trajectories on the phase-space. These simulations create a picture of how the therapeutic relationship might change given different initial conditions of the client and the therapist. These plots allow for a visual representation of the dynamics within the system, and form the basis for predictions within the model.

[4] In fact, if you overlay both Figure 1(c) and Figure 2, the intersection of the client and therapist influence function exactly correlates with the steady states in the phase space of Figure 2.

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Appendix

The variables $T(t)$ and $C(t)$ denote the therapist's and client's behavior scores respectively at time t. The parameters a_T and a_C denote the "inertia" to change, that is, the degree of dependence of the therapist's and client's behavior scores on their previous values respectively, where a_T, $a_C < 0$. The smaller the absolute value of these parameters, the less is the dependency on previous behavior scores. The parameters b_T and b_C denote the therapist's and client's behavior scores respectively when alone. The last parts of equations are the influence functions, which model the effect that the therapist or client has on the other person, that is, they are a function of either the client's or the therapist's behavioral score.

The functions $f_{C®T}(C)$ and $f_{T®C}(T)$ denote the form of the influence of the client on the therapist and the influence of the therapist on the client respectively and $s_{C®T}$ and $s_{T®C}$ denote the strength of those influences. The complete therapy equations are then:

$$dT/dt = a_T T = b_T + s_{C\rightarrow} T f_{C\rightarrow} T(C) \quad (1a)$$

$$dC/dt = a_C C + b_C + s_{T\rightarrow} C f_{T\rightarrow} C(T) \quad (1b)$$

Translating these mathematical symbols into words, these equations state that the rate of change of the behavior state of the therapist and client (dT/dt and dC/dt) at time t is proportional to the sum of their "inertia" to change ($a_T T$ and $a_C C$), their uninfluenced behavior state (b_T and b_C), and the influence from each other ($s_{C\rightarrow T}\ f_{C\rightarrow T}(C)$ and $s_{T\rightarrow C}\ f_{T\rightarrow C}(T)$). In other words, both the therapist and the client begin each session with an initial behavior score, and then these scores change over time based on the influence (or interaction) of the other person over time.

Steady states in the therapeutic relationship. The behavior state of both the therapist (T) and client (C) start from some initial values and both evolve in time. We are very interested in determining where they wind up, as this represents the outcome of the therapy for both the therapist and client. Such steady states, if they exist, must occur where the values of T and C are no longer changing in time, namely where the derivatives $dT/dt = 0$ and $dC/dt = 0$. The special values of T and C where this occurs are called the "critical points." The equations for $dT/dt = 0$ and $dC/dt = 0$ are each called the "nullclines." Since the critical points satisfy the equations of both nullclines, they lie at the intersections of the two nullclines. Therefore, we can easily see the values of these critical points graphically as the intersections of the nullclines on a plot where the behavior state of the therapist is on the vertical axis and the behavior state of the client is on the horizontal axis.

For the first case that we study here, the client and therapist each have an equal strength of influence over each other, although the shape of their influence functions is each different (as described above). Since, for this case, the "inertia" parameters $a_T = a_C = -1$, the nullclines are the same as the influence functions.